

Rational Chebyshev Approximation on Subsets

CHARLES B. DUNHAM

Department of Computer Science, University of Western Ontario, London, Ontario, Canada

In this note we study the closeness of a best Chebyshev approximation by generalized rational functions on a set, to a corresponding best approximation on a subset thereof. Such a problem is of practical interest, as we often determine a best approximation on an interval or rectangle in 2-space, as a limit of best approximations on finite point sets.

Let X be a compact set and let $\{\phi_1, \dots, \phi_n\}$, $\{\psi_1, \dots, \psi_m\}$ be linearly independent subsets of $C(X)$. Define

$$R(A, x) := P(A, x)/Q(A, x) := \frac{\sum_{k=1}^n a_k \phi_k(x)}{\sum_{k=1}^m a_{n+k} \psi_k(x)}.$$

The conventions of Boehm (assuming his dense nonzero property is satisfied) or of Goldstein (stabilized rational functions) ([2], Chapter 9) can be used to give $R(A, x)$ a value when $Q(A, x) = 0$. For Y a compact subset of X , define

$$\|g\|_Y = \sup\{|g(x)| : x \in Y\},$$

$$\mathcal{R}(Y) = \{R(A, \cdot) : \|R(A, \cdot)\|_Y < \infty, \quad Q(A, x) \geq 0 \text{ for } x \in Y, Q(A, \cdot) \not\equiv 0\}.$$

The problem of rational approximation of $f \in C(X)$ on Y , is to choose $r \in \mathcal{R}(Y)$ such that $\|f - r\|_Y$ is minimal. Such an r is called a best approximation to f on Y . It is a consequence of the theory of Boehm or Goldstein (depending on whose conventions we use) that a best approximation exists to all $f \in C(X)$ on Y .

Let $\{X_k\}$ be a given sequence of subsets of X such that for any $x \in X$, there is an $x_k \in X_k$ such that $\{x_k\} \rightarrow x$. Let $R(A_k, \cdot)$ be a best approximation to f on X_k . We obtain in this note several results concerning convergence of $\{R(A_k, \cdot)\}$ to a best approximation to f on X .

For convenience in existence and convergence arguments, we normalize rational functions $R(A, \cdot)$ so that

$$\sum_{k=1}^m |a_{n+k}| = 1. \quad (1)$$

THEOREM 1. *Let r^* be best to f on X . Then*

$$\|f - R(A_k, \cdot)\|_{X_k} \rightarrow \|f - r^*\|_X.$$

Proof. Define

$$\|A\| = \sum_{k=1}^n |a_k|.$$

Suppose that $\{\|A_k\|\}$ is an unbounded sequence. We can then assume without loss of generality, that $\{\|A_k\|\}$ is also a strictly increasing sequence. Define $B_k = A_k/\|A_k\|$; then $\{B_k\}$ is a bounded sequence with a limit point B . Select $z \in X$ such that $P(B, z) \neq 0$. Then there is a closed neighborhood N of z on which $P(B, \cdot)$ does not vanish. Assume that $P(B, z) > 0$. Then

$$\eta = \inf\{P(B, x) : x \in N\}$$

is positive and for all k sufficiently large, $P(B_k, x) > \eta/2$ for $x \in N$. It follows that $\inf\{P(A_k, x) : x \in N\} \rightarrow \infty$ as $k \rightarrow \infty$. For all k sufficiently large we must have a point $x_k \in X_k \cap N$ such that

$$P(A_k, x_k) > 2\|f\|_X \sum_{k=1}^m \|\psi_k\|_X.$$

This implies that $R(A_k, x_k) > 2\|f\|_X$, which is impossible as

$$\|f - R(A_k, \cdot)\|_{X_k} \geq |f(x) - R(A_k, x_k)| > \|f - 0\|_X,$$

contradicting $R(A_k, \cdot)$ being best on X . It follows that $\{\|A_k\|\}$ is bounded (that is, the numerator coefficients of $\{R(A_k, \cdot)\}$ are bounded). The denominator coefficients are bounded by the normalization (1).

$\{A_k\}$ being bounded, has a convergent subsequence, which we assume, without loss of generality, is $\{A_k\}$ itself, with limit A . We claim that $R(A, \cdot)$ is a best approximation to f on X . Suppose the contrary; then there exists a point x and a positive ϵ such that

$$|f(x) - R(A, x)| > \|f - r^*\|_X + \epsilon. \quad (2)$$

The first possibility is that $Q(A, x) = 0$ and $P(A, x) \neq 0$. Let $\{x_k\} \rightarrow x, x_k \in X_k$; then $|R(A_k, x_k)| \rightarrow \infty$, which is impossible. The second possibility is that $Q(A, x) = P(A, x) = 0$. In the theory of Goldstein, $R(A, x)$ can always be defined in this case so that $f(x) - R(A, x) = 0$. In the theory of Boehm, we can find in the neighborhood of x a point at which $Q(A, \cdot)$ does not vanish and for which an inequality of the type (2) holds. Thus, we need only consider the remaining possibility, which is that $Q(A, x) \neq 0$. In this case, let $\{x_k\} \rightarrow x, x_k \in X_k$; then $\{|f(x_k) - R(A_k, x_k)|\} \rightarrow |f(x) - R(A, x)|$ and for all k sufficiently large,

$$|f(x_k) - R(A_k, x_k)| > \|f - r^*\|_X + \epsilon.$$

This contradicts $R(A_k, \cdot)$ being best to f on X_k . Thus, $R(A, \cdot)$ is best and the theorem follows.

It is easily seen that $\{A_k\} \rightarrow A$, $Q(A, x) > 0$ imply that $R(A_k, x) \rightarrow R(A, x)$. An examination of the proof of the theorem gives

COROLLARY 1. *The sequence $\{A_k\}$ has a limit point. For any such limit point A , $R(A, \cdot)$ is a best approximation to f on X , and a subsequence $\{R(A_{k(j)}, \cdot)\}$ of $\{R(A_k, \cdot)\}$ exists converging pointwise to $R(A, \cdot)$ outside the zeros of $Q(A, \cdot)$.*

If $\{A_k\} \rightarrow A$ and $Q(A, \cdot)$ has no zeros, $\{R(A_k, \cdot)\}$ converges uniformly to $R(A, \cdot)$. Again, an examination of the proof of Theorem 1 gives

COROLLARY 2. *Suppose f has a unique best approximation $R(A, \cdot)$ which cannot be represented in the form $R(C, \cdot)$, with $Q(C, \cdot)$ having a zero in X . Then $\{R(A_k, \cdot)\}$ converges uniformly to $R(A, \cdot)$ on X .*

Define

$$S(A) = \{R(A, \cdot) Q(B, \cdot) + P(B, \cdot)\}.$$

THEOREM 2. *Let $R(A, \cdot)$ be best, $Q(A, \cdot) > 0$, and $S(A, \cdot)$ a Haar subspace. Then $R(A, \cdot)$ is the unique best approximation.*

Proof. It is known ([1], p. 164) that $S(A, \cdot)$ being a Haar subspace implies that $R(A, \cdot) = p/q$ is the unique best approximation to f on X among rational functions with positive denominators. Suppose $R(C, \cdot) = s/t$ is also a best approximation, t having a zero. Since

$$\frac{p(x) + s(x)}{q(x) + t(x)} = \begin{cases} \frac{q(x)}{q(x) + t(x)} \cdot \frac{p(x)}{q(x)} + \frac{t(x)}{q(x) + t(x)} \cdot \frac{s(x)}{t(x)} & \text{if } t(x) \neq 0, \\ \frac{p(x)}{q(x)} & \text{if } s(x) = t(x) = 0, \end{cases}$$

$(p + s)/(q + t)$ lies between p/q and s/t . It must therefore be also a best approximation. But it has a positive denominator, contradicting the uniqueness of p/q among such functions.

COROLLARY 3. *Let f have $R(A, \cdot)$ as a best approximation, $Q(A, \cdot) > 0$, and $S(A, \cdot)$ be an $(n + m - 1)$ -dimensional Haar subspace. Then $\{R(A_k, \cdot)\}$ converges uniformly to $R(A, \cdot)$ on X .*

Proof. $S(A, \cdot)$ is $(n + m - 1)$ -dimensional, $R(A, \cdot)$ cannot be represented in the form $R(C, \cdot)$, $Q(C, \cdot)$ with a zero. By Theorem 2, $R(A, \cdot)$ is the unique best approximation. The Corollary follows from Corollary 2.

The theory of this note can be applied to the classical problem of approximation by ratios of polynomials of degree $n-1$ to polynomials of degree $m-1$, on an interval $[a, b]$. Such ratios are customarily restricted to have positive denominators on $[a, b]$; but as any ratio of polynomials bounded on $[a, b]$ can be written in this form, this is no restriction. Assume without loss of generality that $Q(A, \cdot) > 0$ and that $P(A, \cdot)$, $Q(A, \cdot)$ are relatively prime. Let $P(A, \cdot)$, $Q(A, \cdot)$ have exact degrees i, j , respectively. The subspace $S(A)$ is the space of ratios of polynomials of degree $n+m-2-\min\{n-1-i, m-1-j\}$ to $Q(A, \cdot)$, a Haar subspace of dimension $n+m-1-\min\{n-1-i, m-1-j\}$. The hypotheses of Corollary 3 are satisfied if and only if $i=n-1$ or $j=m-1$, that is $R(A, \cdot)$ is nondegenerate, or, equivalently, of maximum degree (in the sense of Rice).

We give an example to show that $\{R(A_k, \cdot)\}$ need not converge uniformly to $R(A, \cdot)$ even if $R(A_k, \cdot)$ is a unique best approximation. Let $X = [-1, 1]$, $f(x) = x$, and $R(A, x) = a_1/(a_2 + a_3 x)$. As f alternates once on $[-1, 1]$, 0 is the unique best approximation to f . Now let $X_k = [-1 + 1/k, 1]$; then f does not alternate once on X_k . Since all elements of $\mathcal{R}(X_k)$ except the zero element are of degree 2, the unique best approximation r_k to f on X_k is characterized by $f - r_k$ alternating at least twice on X_k . If $\{r_k\}$ converged uniformly to 0, f would have to alternate twice. By drawing a diagram it can be seen that $r_k(-1)$ does not converge to zero. This example shows that the dimensionality condition of Corollaries 2 and 3 cannot be deleted.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory." McGraw-Hill, New York, 1966.
2. J. RICE, "The Approximation of Functions," Volume 2. Addison-Wesley, Reading, 1969.